Bounds on complexity of matrix multiplication away from CW tensors

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Complexity of matrix multiplication: ω

Definition of complexity: smallest ω such that for any $\varepsilon > 0$

- the multiplication of $n \times n$ matrices requires $O(n^{\omega+\varepsilon})$ operations
- the **matrix multiplication tensor** has **border rank** $O(n^{\omega+\varepsilon})$

$$\mathrm{Mat}_n \times \mathrm{Mat}_n \to \mathrm{Mat}_n$$

$$M_{\langle n,n,n\rangle} := \sum e^*_{ij} \otimes e^*_{jk} \otimes e_{ik} \in (\mathrm{Mat}_n)^* \otimes (\mathrm{Mat}_n)^* \otimes \mathrm{Mat}_n$$

The naive bound $\omega \leq 3$ has been successively improved:

1969 only 7 operations are required to multiply two 2×2 matrices [Str69]
1987 laser method applied to Strassen's tensor: $\omega < 2.48$ [Str87]
1990 laser method applied to Coppersmith-Winograd's tensor: $\omega < 2.3755$ [CW90]

Conjecture: $\omega = 2$

State of the art

- Stothers, Williams, Le Gall: $\omega < 2.373$ applying laser method to Coppersmith-Winograd (CW) tensor
- Barriers for CW tensors: $\omega > 2.30$
- Barriers for minimal border rank tensors of fixed dimension: $\omega = 2$ cannot be proved [BL20]

Our goal

Explore new families of tensors coming from algebraic structures such that

- have minimal border rank and
- degenerate to a direct sum of large matrix multiplication tensors

Laser method

The **value** of a tensor T is

$$V_\omega(T) := \sup_N \left(\sup \left\{ \sum_{i=1}^{q} (a_ib_ic_i)^{\omega/3} \mid T^{\boxtimes N} \trianglerighteq \bigoplus_{i=1}^{q} M_{\langle a_i,b_i,c_i\rangle} \right\} \right)^{1/N}$$

- Estimate the value of a tensor from subtensors with known value
- Given a tensor of minimal border rank, a lower bound on the value gives an upper bound on ω

Heighest weight vectors...

...of the SL_n-representation of $\mathrm{S}^3(\mathrm{GL}_n)$ [Sey18]

$$T_{HW,m} = 6a_1a_2b_0 + 3a_1^2b_1 + 3a_2^2b_2 + 6\sum_{j=1}^{m}(a_1x_jy_j + a_2x_jz_j) \in \mathrm{S}^3(V)$$

$$V = \langle a_1, a_2\rangle \oplus \langle x_1,\dots,x_m,y_1,\dots,y_m,z_1,\dots,z_m\rangle \oplus \langle b_0,b_1,b_2\rangle$$

has **minimal border rank**. Applying the laser method to the blocking

$$\left(\begin{array}{cc||ccc|ccc|ccc||ccc}
b_1 & b_0 & y_1 & \dots & y_m & x_1 & \dots & x_m & 0 & \dots & 0 & a_2 & a_1 & 0 \\
b_0 & b_2 & z_1 & \dots & z_m & 0 & \dots & 0 & x_1 & \dots & x_m & a_1 & 0 & a_2 \\
\hline\hline
y_1 & z_1 & & & & a_1 & & & a_2 & & & & & \\
\vdots & \vdots & & & & & \ddots & & & \ddots & & & & \\
y_m & z_m & & & & & & a_1 & & & a_2 & & & \\
\hline
x_1 & 0 & a_1 & & & & & & & & & & & \\
\vdots & \vdots & & \ddots & & & & & & & & & & \\
x_m & 0 & & & a_1 & & & & & & & & & \\
\hline
0 & x_1 & a_2 & & & & & & & & & & & \\
\vdots & \vdots & & \ddots & & & & & & & & & & \\
0 & x_m & & & a_2 & & & & & & & & & \\
\hline\hline
a_2 & a_1 & & & & & & & & & & & & \\
a_1 & 0 & & & & & & & & & & & & \\
0 & a_2 & & & & & & & & & & & &
\end{array}\right)$$

gives $\boldsymbol{\omega} < \mathbf{2.45}$ ($m = 7$).

Multiplication maps of smoothable algebras

- Multiplication maps of smoothable algebras are tensors of minimal border rank [BL16]
- Local algebras with Hilbert function $(1, n, 1)$ and $(1, n, 2)$ are smoothable [CEVV09]

$A = \mathbb{C}[a_1,\dots,a_{3m+2}]/I$ is a smoothable algebra with Hilbert function $(1, 3m, 2)$, I ideal generated by

- $a_{3m+1} - a_ia_{m+i}$ for $1 \leq i \leq m$,
- $a_{3m+2} - a_ia_{2m+i}$ for $1 \leq i \leq m$,
- products a_ia_j, $1 \leq i \leq j \leq 3m+2$ not above.

The multiplication map of A is a tensor T_A

$$\left(\begin{array}{c||ccc|ccc|ccc||cc}
\lambda_0 & \lambda_1 & \dots & \lambda_m & \lambda_{m+1} & \dots & \lambda_{2m} & \lambda_{2m+1} & \dots & \lambda_{3m} & \lambda_{3m+1} & \lambda_{3m+2} \\
\hline\hline
\lambda_1 & & & & \lambda_{3m+1} & & & \lambda_{3m+2} & & & & \\
\vdots & & & & & \ddots & & & \ddots & & & \\
\lambda_m & & & & & & \lambda_{3m+1} & & & \lambda_{3m+2} & & \\
\hline
\lambda_{m+1} & \lambda_{3m+1} & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
\lambda_{2m} & & & \lambda_{3m+1} & & & & & & & & \\
\hline
\lambda_{2m+1} & \lambda_{3m+2} & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
\lambda_{3m} & & & \lambda_{3m+2} & & & & & & & & \\
\hline\hline
\lambda_{3m+1} & & & & & & & & & & & \\
\lambda_{3m+2} & & & & & & & & & & &
\end{array}\right)$$

$$A \simeq \langle a_{3m+1}, a_{3m+2}\rangle \oplus \langle a_1,\dots,a_{3m}\rangle \oplus \langle a_0\rangle$$

The laser method yields $\boldsymbol{\omega} < \mathbf{2.431}$ ($m = 4$).

Special case: **multiplication maps of monomial algebras**

$$A = \mathbb{C}[x]/(x^2)$$

$$A_3 := A \otimes A \otimes A \simeq \mathbb{C}[x,y,z]/(xyz)^\perp$$

The multiplication table of A_3 is the tensor T_{A_3}

$$\left(\begin{array}{c|ccc|ccc|c}
a & b & c & d & e & f & g & h \\
\hline
b & 0 & e & f & 0 & 0 & h & 0 \\
c & e & 0 & g & 0 & h & 0 & 0 \\
d & f & g & 0 & h & 0 & 0 & 0 \\
\hline
e & 0 & 0 & h & 0 & 0 & 0 & 0 \\
f & 0 & h & 0 & 0 & 0 & 0 & 0 \\
g & h & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
h & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \quad (1)$$

$\omega < 2.56$ but T_{A_3} degenerates to a Coppersmith-Winograd tensor!

Conclusions

- There exist families of tensors away from CW that already give better bounds than Strassen's tensor
- Laser method is not optimal to analyse (1), new methods should be explored

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