Bounds on complexity of matrix multiplication away from CW tensors

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Complexity of matrix multiplication: $\omega$

Definition of complexity: smallest $\omega$ such that for any $\epsilon > 0$
- the multiplication of $n \times n$ matrices requires $O(n^{\omega+\epsilon})$ operations
- the matrix multiplication tensor has border rank $O(n^{\omega+\epsilon})$

$$M_{(n,n,n)} := \sum_{j=1}^{\omega} e_j^* \otimes e_j \otimes e_j \in (\text{Mat}_n)^* \otimes (\text{Mat}_n)^* \otimes \text{Mat}_n$$

The naive bound $\omega \leq 3$ has been successively improved:
- 1969: only 7 operations are required to multiply two $2 \times 2$ matrices
- 1987: laser method applied to Strassen's tensor: $\omega < 2.48$ [Str87]
- 1990: laser method applied to Coppersmith-Winograd's tensor: $\omega < 2.3755$ [CW90]

Conjecture: $\omega = 2$

State of the art
- Stothers, Williams, Le Gall: $\omega < 2.373$ applying laser method to Coppersmith-Winograd (CW) tensor
- Barriers for CW tensors: $\omega > 2.30$
- Barriers for minimal border rank tensors of fixed dimension: $\omega = 2$ cannot be proved [BL20]

Laser method

The value of a tensor $T$ is
$$V(T) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \left( \sum_{k=1}^{n} T_{ijk} \right) \right)$$

- Estimate the value of a tensor from subtensors with known value
- Given a tensor of minimal border rank, a lower bound on the value gives an upper bound on $\omega$

Our goal

Explore new families of tensors coming from algebraic structures such that
- have minimal border rank and
- degenerate to a direct sum of large matrix multiplication tensors

Multiklick maps of smoothable algebras

- Multiplication maps of smoothable algebras are tensors of minimal border rank [BL16]
- Local algebras with Hilbert function (1,1,1) and (1,2,2) are smoothable [CEVV09]

$$A = \mathbb{C}[z_1, \ldots, z_m]/I_G$$

is a smoothable algebra with Hilbert function (1,3m,2), $I_G$ ideal generated by
$$a_{3m+1} = a_{1m+1}$$
$$a_{3m+2} = a_{2m+1}$$

products $a_ia_j$, $1 \leq i < j \leq 3m+2$ not above.

The multiplication map of $A$ is a tensor $T_A$

### Special case: multiplication maps of monomial algebras

$$A := A \otimes A \simeq \mathbb{C}[x,y]/(xyz)^2$$

The multiplication table of $A$ is the tensor $T_A$

$$\begin{array}{cccccc}
A_{00} & A_{01} & A_{02} & A_{10} & A_{11} & A_{12} \\
A_{20} & A_{21} & A_{22} & A_{30} & A_{31} & A_{32} \\
A_{40} & A_{41} & A_{42} & A_{50} & A_{51} & A_{52} \\
\end{array}$$

The laser method yields $\omega < 2.365$ but $T_A$ degenerates to a Coppersmith-Winograd tensor!

References

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