Towards a new spectral system combining Serre and Eilenberg-Moore Spectral Sequences

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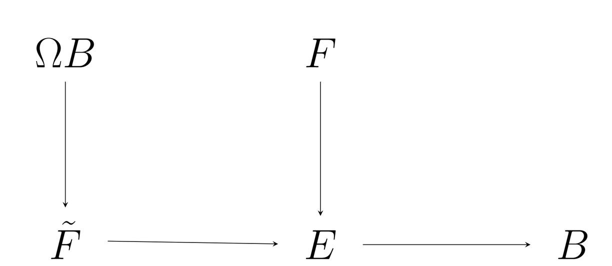
Abstract

The work we are presenting consists in the definition of a **new spectral system** from a fibration of simplicial sets. To do so, we construct a new fibration, which is built from the initial one using the loop space of the base and the universal bundle. Then, we consider **Serre** and **Eilenberg-Moore homology spectral sequences**, both converging to (something similar) to the fiber. These are given by two **filtrations** and built from a Cobar complex. We take them simultaneously to define the new spectral system, with the intention of studying their similarities and their relations. In particular, we apply the technique of **effective homology** and we have the purpose of making computations with the **Kenzo module for spectral systems**.

Approach to the problem

We consider a **principal fibration** of simplicial sets $E \longrightarrow B$ with structural group F, and given by a **twisted cartesian product** $\tau : B_* \longrightarrow F_{*-1}$ such that $E \cong F \times_\tau B$. If we take the loop space of B (by means of the $\mathcal{G}$ construction in [3], section 26) there exists a twisting operator $\tau_1 : B_* \longrightarrow \Omega B_{*-1}$ such that the space $\Omega B \times_{\tau_1} B$ is contractible. This allows us to construct a **second fibration** $\Omega B \longrightarrow \tilde{F} \cong \Omega B \times_{\tilde{\tau}_1} E \longrightarrow E$, where $\tilde{\tau}_1$ is the composition of the projection $E \longrightarrow B$ with τ_1.



Our goals are:

- See how similar F and $\tilde{F}$ are.
- Build a spectral system using both fibrations.
- Make computations with the Kenzo computer algebra program ([1]).

Effective homology

One of the objectives of our work is to make computations, so we may be interested in finitely generated complexes with computable differentials. This kind of complexes are called **effective**. However, our complexes may not be effectives themselves, but sometimes we have chain equivalences that preserve the homology and allow us to carry our computations. These sets of equivalences are called **reductions**.

$$C_* \Longrightarrow D_* \qquad\qquad h \circlearrowright C_* \underset{g}{\overset{f}{\rightleftarrows}} D_*$$

A chain of reductions (in any direction) is called a **strong chain equivalence**. If a chain complex has an equivalence with an effective one, we say that it has **effective homology**.

$$C_* \Longleftarrow D_* \Longrightarrow E_*$$

In our case, we will suppose that we have effective homology for B and E. By [4], we have so for the loop space of B whenever it is 1-reduced (that is, it has only one 0-simplex and no non-degenerate 1-simplexes). We suppose that from now on.

By the contractibility of $\Omega B \times_{\tau_1} B$, there exists a reduction from $C_*(\Omega B \times_{\tilde{\tau}_1} E)$ to $C_*(F)$. In consequence, it makes sense to consider

- The **first Eilenberg-Moore homology spectral sequence** for $F \to E \to B$.
- The **Serre homology spectral sequence** for $\Omega B \to \tilde{F} \to E$.

As both of them converge to the homology of F, we can try to compare them and see if we can combine them.

Filtrations and spectral sequences

One natural way for building spectral sequences consists in taking a **filtration** over $\mathbb{Z}$ of a chain complex $C_* = (C_n, d_n)$ (in our case, the associated one to a simplicial set). In other words, we part from a chain of subcomplexes $\ldots \subseteq F_p C_* \subseteq F_{p+1} C_* \subseteq \ldots$ and we define the collection

$$E^r_{p,q} = \frac{F_p C_n \cap d^{-1}(F_{p-1} C_{n-1})}{d(F_{p+r-1} C_{n+1}) + F_{p-1} C_n}$$

with the induced differential maps.

The Eilenberg-Moore spectral sequence

To define the Eilenberg-Moore spectral sequence over $F \to E \to B$ we take the **cobar construction** $\mathrm{Cobar}^{C_*(B)}(\mathbb{Z}, C_*(E))$, which is isomorphic to a perturbed complex $\mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z}) \otimes_{t_1} C_*(E)$. We define the filtration

$$F^{EM}_k(\mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z}) \otimes_{t_1} C_*(E))_n := \bigoplus_{p \leqslant k} (C_*(B)^{\otimes -p} \otimes_{t_1} C_*(E))_q.$$

The associated spectral sequence is given in the second page by $\mathrm{Cotor}^{H_*(B)}(\mathbb{Z}, H_*(E))$ and **converges to the homology groups of** F.

The Serre spectral sequence

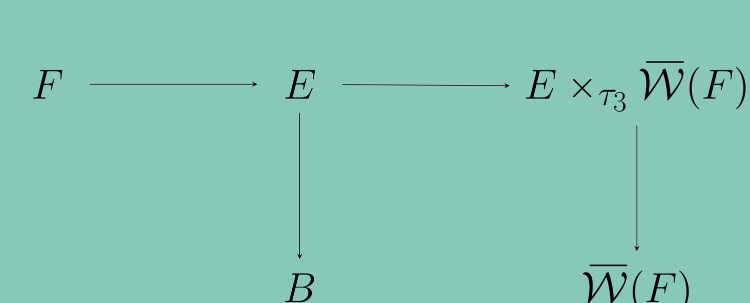
For the fibration $\Omega B \to \tilde{F} \to E$, we can build the Serre spectral sequence parting from the complex $C_*(\Omega B \times_{\tilde{\tau}_1} E)$, which by the **Eilenberg-Zilber Theorem** is isomorphic to $C_*(\Omega B) \otimes_{t_2} C_*(E)$. The key point is that we have an equivalence between $C_*(\Omega B)$ and $\mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z})$ (see [4]). So, we can consider the complex $C_*(\Omega B) \otimes_{t_2} C_*(E)$, and define

$$F^S_k(\mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z}) \otimes_{t_2} C_*(E))_n := \bigoplus_{l \leqslant k} \mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z})_{n-l} \otimes_{t_2} C_*(E)_l.$$

The associated spectral sequence is given in its second page by $H_*(E; H_*(\Omega B))$, and **converges to the homology groups of** $\tilde{F}$.

The dual case

Instead of the loop space of the base, we can consider the **classifying space** of the fiber, and work with the diagram on the right. This case is **dual**, and we could follow similar steps with the Eilenberg-Moore spectral sequence **for the base** of the fibration.



Spectral systems

Spectral systems generalize spectral sequences, and, in particular, can also be defined by a filtration $\{F_i\}$ (this time indexed **over any poset** I). We define each term of the system by

$$S[z,s,p,b] := \frac{F_p \cap d^{-1}(F_z)}{d(F_b) + F_s}$$

for z, s, p and b in I (taking again the induced differentials). In particular, we can combine $\mathbb{Z}$-filtrations to obtain $D(\mathbb{Z}^m)$-filtrations (see [2] for details).

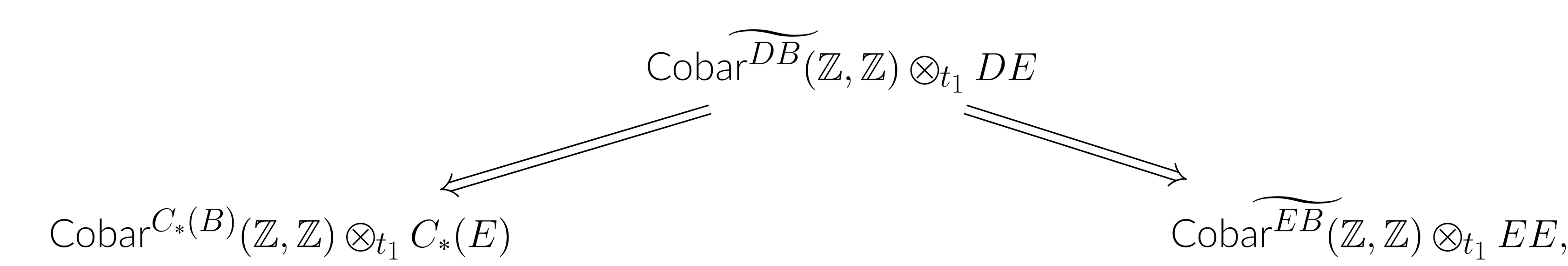
The new spectral system

As the complexes $\mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z}) \otimes_{t_1} C_*(E)$ and $\mathrm{Cobar}^{C_*(B)}(\mathbb{Z},\mathbb{Z}) \otimes_{t_2} C_*(E)$ have the same underlying modules, we can **define both filtrations in one of them** (for example the first one), and consider the associated spectral system. We define the $D(\mathbb{Z}^2)$-filtration

$$F_p := \sum_{(k_1,k_2) \in p} F^S_{k_1} \cap F^{EM}_{k_2}.$$

We have proved the following results:

- By the effective homology for the base and the total space, we have effective homology for the cobar complexes and the twisted tensor products (see [4]). **We can extend our filtrations to the effective complexes and define similar spectral systems there.**



- By the compatibility of the maps of the reductions quoted above with the generalized filtration, these **induce isomorphisms on spectral systems**, whenever $(s_1, s_2) \geqslant (z_1 + 1, z_2)$ and $(b_1, b_2) \geqslant (p_1 + 1, p_2)$.

Further work

We are currently interested in implementing this spectral system in **Kenzo**, as well as the dual case. The objective is, as both filtrations converge to the same homology groups, **study how they interact in the spectral system**, and observe interesting examples in Kenzo. Moreover, we are willing to study the definitions above for other complexes, and the **relations between the perturbations** for $\otimes_{t_1}$ and $\otimes_{t_2}$. Also, it may be interesting to look at the **Puppe sequence** and try to iterate this construction to higher loop and classifying spaces.

References

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