Generating functions of degrees of Kalman varieties

Luca Sodomaco, Aalto University, Espoo, Finland. Email: luca.sodomaco@aalto.fi
joint work with Zahra Shahidi and Emanuele Ventura

Aalto University
School of Science

Singular vector tuples and eigenvectors of tensors

Given $\boldsymbol{\omega}, \mathbf{n} \in \mathbb{N}^k$, we denote by $\mathrm{Sym}^{\boldsymbol{\omega}}\mathbb{C}^{\mathbf{n}}$ the vector space $\bigotimes_{i=1}^k \mathrm{Sym}^{\omega_i}\mathbb{C}^{n_i}$ of complex **partially symmetric tensors** of format $n_1^{\times \omega_1} \times \cdots \times n_k^{\times \omega_k}$. Every element of $\mathrm{Sym}^{\boldsymbol{\omega}}\mathbb{C}^{\mathbf{n}}$ is a linear combination of **decomposable** tensors $\mathbf{x}_1^{\omega_1} \otimes \cdots \otimes \mathbf{x}_k^{\omega_k}$ for some vectors $\mathbf{x}_i \in \mathbb{C}^{n_i}$.

Definition. Given a tensor $T \in \mathrm{Sym}^{\boldsymbol{\omega}}\mathbb{C}^{\mathbf{n}}$, a **singular vector tuple (svt)** of T is a k-tuple $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of nonzero vectors $\mathbf{x}_i \in \mathbb{C}^{n_i}$ such that

$$\mathrm{rank}\begin{pmatrix} T \cdot \mathbf{x}_1^{\omega_1} \otimes \cdots \otimes \mathbf{x}_i^{\omega_i - 1} \otimes \cdots \otimes \mathbf{x}_k^{\omega_k} \\ \mathbf{x}_i \end{pmatrix} \leq 1 \quad \forall i \in [k],$$

where the dot in the first row of the matrix denotes tensor contraction.

Singular vector tuples generalize singular vector pairs and eigenvectors of matrices.

Kalman varieties of tensors

The **Kalman variety** is the variety of tensors possessing a svt with each entry on a fixed subspace.

Definition. For every $i \in [k]$, let $L_i \subset \mathbb{P}(\mathbb{C}^{n_i})$ be a projective subspace and consider the product $L = L_1 \times \cdots \times L_k$. The **Kalman variety** associated with L is

$$\kappa_{\mathbf{n}}^{\boldsymbol{\omega}}(L) := \{T \in \mathbb{P}(\mathrm{Sym}^{\boldsymbol{\omega}}\mathbb{C}^{\mathbf{n}}) \mid T \text{ has a svt } (\mathbf{z}_1, \ldots, \mathbf{z}_k) \in L\}$$

This variety was originally introduced by Ottaviani and Sturmfels in [5] in the matrix case ($k = 2$) and when $L = L_1 \times \mathbb{P}(\mathbb{C}^{n_2})$. They determined codimension, degree and studied its singular locus. Thereafter, Sam [7] and Huang [3] determined their defining equations in the same setting.
Ottaviani and Shahidi [4] proved the irreducibility of Kalman varieties for tensors when $L = L_1 \times \mathbb{P}(\mathbb{C}^{n_2}) \times \cdots \times \mathbb{P}(\mathbb{C}^{n_k})$, and computed their codimensions and degrees.

Codimensions and degrees of Kalman varieties

Theorem. *We assume each L_i is real of codimension δ_i. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$.*

i) The Kalman variety $\kappa_{\mathbf{n}}^{\boldsymbol{\omega}}(Z)$ is irreducible of codimension $\delta := \sum_{i=1}^k \delta_i$.

ii) The degree of $\kappa_{\mathbf{n}}^{\boldsymbol{\omega}}(Z)$ is the coefficient $d(\mathbf{n}, \boldsymbol{\delta}, \boldsymbol{\omega})$ of the monomial $h^\delta \prod_{i=1}^k t_i^{n_i - \delta_i - 1}$ in the polynomial

$$\prod_{i=1}^k \frac{[(\widehat{t}_i + h)^{n_i} - t_i^{n_i}]}{(\widehat{t}_i + h) - t_i}, \quad \widehat{t}_i := \left(\sum_{j=1}^k \omega_j t_j\right) - t_i .$$

Example. In the symmetric case $k = 1$, $\mathbf{n} = (n)$, $\boldsymbol{\delta} = (\delta)$, $\boldsymbol{\omega} = (\omega)$, we have

$$d(n, \delta, \omega) = \sum_{j=0}^{n-\delta-1} \binom{\delta + j}{j} (\omega - 1)^j .$$

The values $d(\mathbf{n}, \boldsymbol{\delta}, \boldsymbol{\omega})$ behave in a similar way to the numbers of svt's of a general tensor in $\mathbb{P}(\mathrm{Sym}^{\boldsymbol{\omega}}\mathbb{C}^{\mathbf{n}})$, computed by Ottaviani and Friedland [2].

Generating function of the degrees $d(\mathbf{n}, \boldsymbol{\delta}, \boldsymbol{\omega})$

The last theorem establishes the **enumerative** nature of the subject. When enumerative structures appear, it is a natural problem to determine a generating function whose coefficients are the counted quantities. Generating functions are tremendously useful tools to have a global picture of the enumerated objects.
For example, Zeilberger [1] found a generating function for Friedland-Ottaviani's formula for the number of svt's of a tensor.
Our main result provides a generating function of the degrees of a family of Kalman varieties.

Theorem. *If $\boldsymbol{\delta} = (\delta, 0, \ldots, 0)$ for some $\delta \geq 0$, the generating function for the degrees $d(\mathbf{n}, \delta, \boldsymbol{\omega})$ is*

$$\sum_{\mathbf{n} \in \mathbb{N}^k} \sum_{\delta=0}^{\infty} d(\mathbf{n}, \delta, \boldsymbol{\omega})\, \mathbf{x}^{\mathbf{n}} y^\delta = \frac{1}{H_{\boldsymbol{\omega}}(\mathbf{x}, y)} \prod_{i=1}^k \frac{x_i}{1 - x_i}$$

where

$$H_{\boldsymbol{\omega}}(\mathbf{x}, y) := -y\, x_1 \prod_{i=2}^k (1 + x_i) + \prod_{i=1}^k (1 + x_i) - \sum_{j=1}^k \omega_j x_j \prod_{i \neq j} (1 + x_i) .$$

Asymptotic behaviour of the degrees $d(\mathbf{n}, \boldsymbol{\delta}, \boldsymbol{\omega})$

We study the degree $d(\mathbf{n}, \boldsymbol{\delta}, \boldsymbol{\omega})$ for $\mathbf{n} = n\mathbf{1}$, $\boldsymbol{\delta} = (\delta, 0, \ldots, 0)$ and $\boldsymbol{\omega} = \omega\mathbf{1}$ for $n \to \infty$.
The next formula generalizes Pantone's asymptotic formula [6] for the number of svt's of a tensor.

Theorem. *Consider the degree $d(n, \delta, \omega)$ of the Kalman variety $\kappa_{n\mathbf{1}}^{\omega\mathbf{1}}(Z)$. Assume that either $k \geq 3$ or $k = 2$ and $\omega \geq 2$. Then asymptotically, for $n \to +\infty$,*

$$d(n, \delta, \omega) = \frac{(\omega k - 1)^{k-1}}{(2\pi)^{\frac{k-1}{2}} (\omega k)^{\frac{k-2}{2}} (\omega k - 2)^{\frac{3k-1}{2}}} \left(\frac{\omega k}{\omega k - 1}\right)^\delta \frac{(\omega k - 1)^{kn}}{n^{\frac{k-1}{2} - \delta}} \left[1 + O\left(\frac{1}{n}\right)\right] .$$

Example. When $\omega = 1$, we obtain the following $O(1/n)$-approximations for $d(n, \delta, 1)$:

$$d(n, \delta, 1) \approx \begin{cases} \frac{2}{\sqrt{3\pi}} \left(\frac{3}{2}\right)^\delta 8^n\, n^{\delta - 1} & \text{if } k = 3 \\ \frac{27}{2^9 \pi \sqrt{\pi}} \left(\frac{4}{3}\right)^\delta 81^n\, n^{\delta - \frac{3}{2}} & \text{if } k = 4 . \end{cases}$$

References

[1] Shalosh B. Ekhad and Doron Zeilberger. *On the number of singular vector tuples of hyper-cubical tensors.* The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, 2016.

[2] Shmuel Friedland and Giorgio Ottaviani. *The number of singular vector tuples and uniqueness of best rank-one approximation of tensors.* Found. Comput. Math., 14(6):1209-1242, 2014.

[3] Hang Huang. *Equations of Kalman varieties.* To appear in Proc. Amer. Math. Soc., arXiv:1707.08699, 2017.

[4] Giorgio Ottaviani and Zahra Shahidi. *Tensors with eigenvectors in a given subspace.* To appear in Rend. Circ. Mat. Palermo, arXiv:2010.03843, 2020.

[5] Giorgio Ottaviani and Bernd Sturmfels. *Matrices with eigenvectors in a given subspace.* Proc. Amer. Math. Soc., 141(4):1219-1232, 2013.

[6] Jay Pantone. *The asymptotic number of simple singular vector tuples of a cubical tensor.* Online J. Anal. Comb., (12):11, 2017.

[7] Steven V Sam. *Equations and syzygies of some Kalman varieties.* Proc. Amer. Math. Soc., 140(12):4153-4166, 2012.

[8] Luca Sodomaco. *The product of the eigenvalues of a symmetric tensor.* Linear Algebra Appl., 554:224-248, 2018.