Climbing The Wall: ML-Degrees for Nets of Conics

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Background and Motivation

In 1977, Charles Terence Clegg Wall classified all distinct orbits, with respect to coordinate change, of real and complex *nets of conics*. A net of conics is a 3-dimensional linear space of symmetric matrices, which we may also view as a linear space of 3-variate homogeneous quadratic polynomials. As a consequence of Wall's work, all linear subspaces of $\mathbb{S}^3$, the set of 3×3 symmetric matrices, could be classified. In constrast, the study of *pencils*, the 2-dimensional analogue of nets, was classic already 50 years before, going back to work by Weierstrass and C. Segre. This characterization is in general a very difficult problem. Not much is known about nets in higher dimensions. In our article "maximum likelihood estimation for nets of conics" [1], to appear in Le Matematiche, we study Wall's work from the perspective of algebraic statistics.

Our setup is as follows: we study the problem of maximum likelihood estimation, where we view nets of conics as linear concentration models of Gaussian distributions on $\mathbb{R}^3$. The *maximum likelihood degree (ML-degree)* of a real linear space $\mathcal{L} \subset \mathbb{S}^n$ is the number of complex critical points of the log-likelihood function

$$\begin{aligned} \ell_S : \mathcal{L} &\longrightarrow \mathbb{R}, \\ M &\longmapsto \log\det(M) - \operatorname{trace}(SM) \end{aligned}$$

for a generic matrix $S \in \mathbb{S}^n$. Importantly, this function is strictly concave. We should think of S as the sample covariance matrix, which is naturally "random" and therefore generic. The ML-degree $\operatorname{mld}(\mathcal{L})$ tells us about the algebraic complexity of finding the best estimate. It tells us how hard the problem is.

Algebraic geometry enters the picture when we consider the *score equations* of ℓ_S, which we get by putting the partial derivatives to zero. Let $A_1, \ldots, A_m$ be linearly independent matrices that span our concentration model $\mathcal{L}$, so that $\mathcal{L} = \{\sum \lambda_i A_i : \lambda_i \in \mathbb{C}\}$. The *reciprocal variety* $\mathcal{L}^{-1}$ is defined as the Zariski closure of the set of matrices M^{-1} for invertible $M \in \mathcal{L}$. The reciprocal variety is well-defined as long as $\mathcal{L}$ is *regular*, meaning it contains an invertible matrix.

The *polar net* $\mathcal{L}^\perp$ is the set of matrices N such that $N \bullet M = 0$ for all $M \in \mathcal{L}$, where $\bullet$ denotes the trace operator.

The score equations are for $i = 1, \ldots, m$,

$$(\ell_S(M))'_{A_i} = \nabla \ell(M) \bullet A_i = (M^{-1} - S) \bullet A_i = 0.$$

As a direct consequence for a generic S,

$$\operatorname{mld}(\mathcal{L}) = \#\big((\mathcal{L}^{-1} - S) \cap \mathcal{L}^\perp\big).$$

The variety $\big((\mathcal{L}^{-1} - S) \cap \mathcal{L}^\perp\big)$ is finite for generic S and its cardinality is given by its degree. By simulating a random rational matrix, we can empirically find the ML-degree using software like `Macalauy2` [4]. Formulas for the ML-degree of pencils can be found in [3].

Wall's Classification

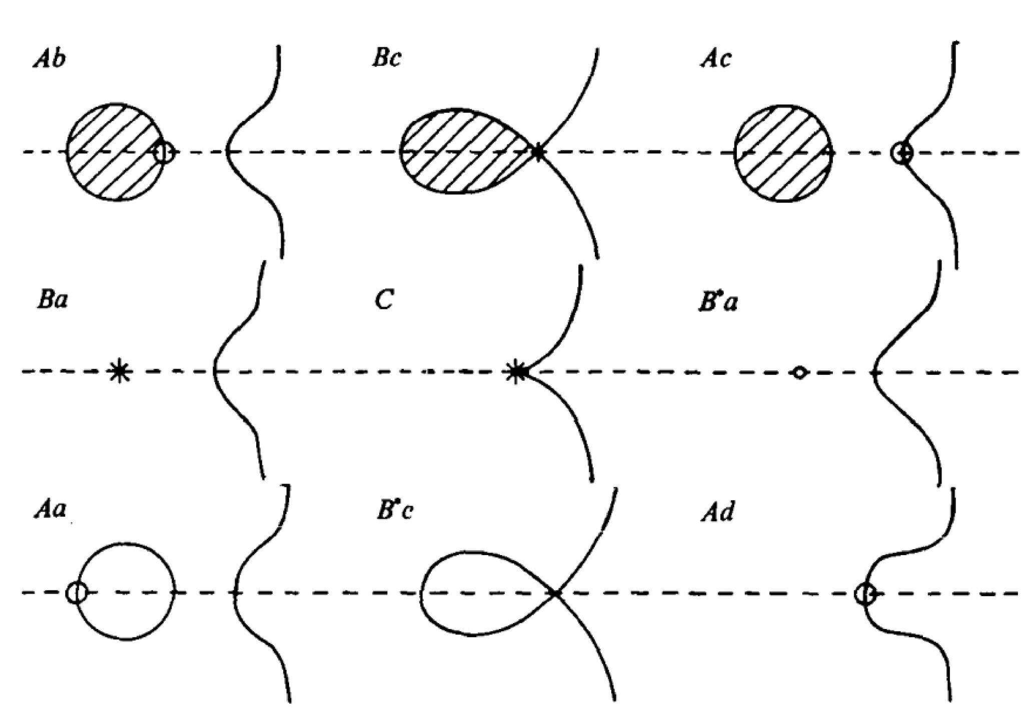

Figure 1: Wall's types A, B, B^* and C divided into different real subtypes. The nine images show the nets intersected with the determinental hypersurface, which is the black curve. The shaded areas correspond to positive definite matrices in the nets. The image is taken from [5].

Wall found that any complex net of conics can be categorized as one of 15 geometric types that Wall refers to as A, B, B^*, C, D, D^*, E, E^*, F, F^*,G, G^*, H I, I^*. His main tool in distinguishing those types is the cubic discriminant curve obtained by intersecting the net $\mathbb{P}\mathcal{L}$ with the determinantal hypersurface in $\mathbb{P}\mathbb{S}^3$ [5], see Figure 1. Type A is a 1-dimensional family of orbits under coordinate change and the other types consist of one orbit. By coordinate change we mean the *congruence* action $\mathcal{L} \mapsto A^T \mathcal{L} A$ by an invertible matrix A.

The polar net of a net of type B is of type B^*. Similarly, the types D and D^*, E and E^*, F and F^*, as well as G and G^* are polar to each other. We note that the types A, C and H are self-polar in that sense.

The Reciprocal Varieties

We study the reciprocal varieties of nets of conics, which are rational surfaces in $\mathbb{P}^5$, because of the connection between their degree and the ML-degree. It turns out to be sufficient to study the reciprocal variety of one representative per congruence class of nets. These varieties of regular nets of conics are closely related to the Veronese surface in $\mathbb{P}^5$, which is the image of the embedding

$$\begin{aligned} \nu : \quad & \mathbb{P}^2 \longrightarrow \mathbb{P}^5, \\ (\alpha : \beta : \gamma) &\longmapsto (\alpha^2 : \beta^2 : \gamma^2 : \alpha\beta : \alpha\gamma : \beta\gamma). \end{aligned}$$

Theorem: *For every net $\mathcal{L}$ the reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$ is described as projections from the Veronese surface according to the following table:*

	A B B^* C D D^* E E^* F F^* G G^* H
Blue	Projectively equivalent to the Veronese surface
Orange	Of degree 3 and is the projection of the Veronese surface from a point
Green	Of degree 2 and is the projection of the Veronese surface from a line
Red	Projection of the Veronese surface from a plane

ML-Degrees

Type	A	B	B^*	C	D	D^*	E	E^*	F	F^*	G	G^*	H
$\deg \mathbb{P}\mathcal{L}^{-1}$	4	3	4	3	2	4	1	4	2	2	1	2	1
$\operatorname{mld} \mathcal{L}$	4	3	3	2	2	2	1	1	0	1	0	0	0

Table 1: We list $\deg \mathbb{P}\mathcal{L}^{-1}$ and $\operatorname{mld} \mathcal{L}$ for regular nets.

Our main tool here is the study of the *ML-base locus* which is the intersection of the reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$ with the polar net $\mathbb{P}\mathcal{L}^\perp$ defined via the trace pairing.

A formula for the ML-degree of a linear space $\mathcal{L} \subset \mathbb{S}^n$ in terms of Segre classes of its ML-base locus is given in [2]. We only use the following special cases:

Proposition: [2] *The ML-degree of a linear space $\mathcal{L} \subset \mathbb{S}^n$ is at most the degree of its reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$. This is an equality if and only if the ML-base locus $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^\perp$ is empty. Moreover, for a linear space $\mathcal{L} \subset \mathbb{S}^n$ whose ML-base locus is finite and consists only of smooth points of $\mathbb{P}\mathcal{L}^{-1}$, we have*

$$\operatorname{mld}(\mathcal{L}) = \deg(\mathbb{P}\mathcal{L}^{-1}) - \deg(\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^\perp),$$

where the degree of the ML-base locus is its scheme-theoretic degree (i.e., the constant coefficient of its Hilbert polynomial).

It is sufficient to compute the ML-base locus for type A and one representative per each other type, which can be done by hand or in `Macaulay2`. Together with our result on the reciprocal varities we obtain our main theorem.

Theorem: *The ML-degree of a regular net of conics depends only on its type, as defined by Wall. All ML-degrees are listed in Table 1.*

References:

[1] Stefan Dye, Kathlén Kohn, Felix Rydell and Rainer Sinn: *Maximum likelihood estimation for nets of conics*, arXiv preprint arXiv:2011.08989 (2021).

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[3] Claudia Fevola, Yelena Mandelshtam, and Bernd Sturmfels: *Pencils of Quadrics: Old and New*, arXiv preprint arXiv:2009.04334 (2020).

[4] Daniel R. Grayson and Michael E. Stillman: *Macaulay2, a software system for research in algebraic geometry*, available at `www.math.uiuc.edu/Macaulay2/`

[5] C.T.C. Wall: *Nets of conics*, Mathematical Proceedings of the Cambridge Philosophical Society **81** (1977) 351–364.