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The complexity of risk measurement

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Risk and Ambiguity vs Uncertainty

Uncertainty[1] is the general lack of sureness. Uncertainty is a status that can be better (partially) understood thanks to the analysis of the chaotic situation.

Risk is defined as a quantifiable uncertainty. Typically the potential outcomes can be described in a set of scenarios in which a probability measure is given. The study of risks reduces the uncertainty into risk management, thanks to risk measures.

Ambiguity is the context in which the scenarios are known, but no referent probability is possible to be identified precisely. In this case, risk analysis needs to be coupled with some form of robustness.

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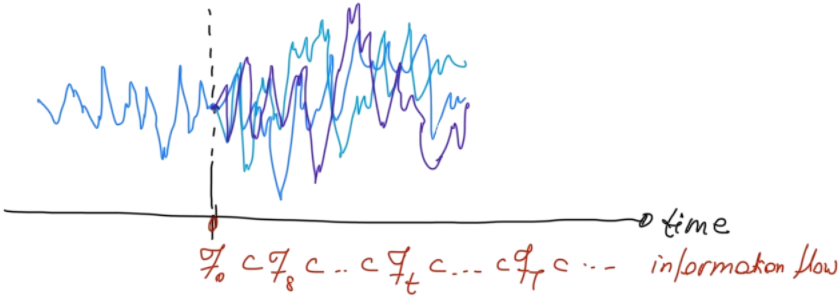
[1] Knight (1921), Ellsberg (1961),... Riedel (2019,2021).

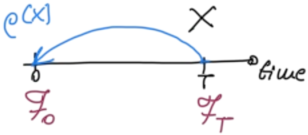
Past, present, future and information

In time dynamics,

- the future scenarios are represented by $(\Omega, \mathcal{F}, P)$.
- The information flow builds up in time $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $s \leq t$.
- For all t, $\mathcal{X}_t$ represents $\mathcal{F}_t$-random variables.



Static risk measures



A static risk measure is a mapping

$$\rho : \mathcal{X}_T \longrightarrow \mathcal{X}_0 \quad (= \mathbb{R})$$

with some properties:

1. monotone: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$
2. translation invariant: if $m \in \mathcal{X}_0$, then $\rho(X + m) = \rho(X) - m$
3. normalized: $\rho(0) = 0$
4. positive homogeneous: for $\lambda > 0$, then $\rho(\lambda X) = \lambda \rho(X)$
5. sub-additive: $\rho(X + Y) \leq \rho(X) + \rho(Y)$
6. **convex**: for $\lambda \in [0, 1]$, then $\rho\Big(\lambda X + (1-\lambda) Y\Big) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$
7. law invariant: if $\mathcal{L}(X) = \mathcal{L}(Y)$, then $\rho(X) = \rho(Y)$

A **monetary evaluation** of an admissible risk: $\rho(X) = inf\{m : \; m + X \in \mathcal{A}\}$.

Some notable cases of large use[2]

- **Value-at-Risk** $\alpha \in (0, 1)$:

$$VaR_\alpha(X) = -q_\alpha^+ = -inf\{x : F_X(x) > \alpha\}$$

VaR is monotone, translation invariant, normalised, positive homogeneous, but not sub-additive (hence VaR penalises diversification). Also, no magnitude.

- **Conditional/average VaR (expected shortfall)** $\alpha \in (0, 1)$:

$$CVaR_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 Var_u(X) du$$

This is a coherent risk measure.

- **Entropic risk measure** $\theta > 0$:

$$\rho^\theta(X) = \frac{1}{\theta} \log E\left[e^{\theta X}\right] = \sup_{Q \in \mathcal{M}_1} \left\{ E_Q\left[X\right] - \frac{1}{\theta} H(Q|P) \right\}$$

relative entropy $H(Q|P) := E\left[\frac{dQ}{dP} log \frac{dQ}{dP}\right]$. Convex, but not coherent

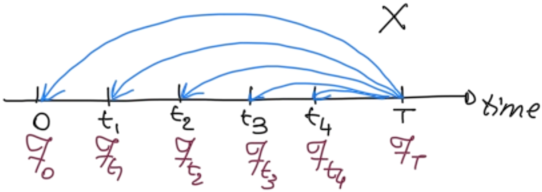
[2] Banking - Basil I, II, III regulatory framework for risk management now with credit, market, and operational risk. Insurance and reinsurance - Solvency I, II prudential regime framework

Dynamic risk measures

Dealing with phenomena in time, also risk assessment has to follow.

A dynamic risk measure is a family of individual risk measures $(\rho_t)_{0 \leq t \leq T}$:

$$\rho_t : \mathcal{X}_T \longrightarrow \mathcal{X}_t$$



Note: Convex risk measures have a convex dual representation, which opens connection with convex analysis.

Dynamic risk measures and BSDEs

Considering the future uncertainty be of Gaussian nature, then Brownian motion can be taken as noise. The information flow is associated to the noise. The random variables have moments, $L^p(P)$ spaces.

Characterisation of rm in terms of BSDEs.[3] Dynamic risk measures are associated to BSDEs (= Backward Stochastic Differential Equations):

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s$$

The process $(Y_t)_t$ in the solution $(Y_t, Z_t)_{t\in[0,T]}$ is regarded as an operator depending on the **driver** g and evaluated at $X \in L^2(\mathcal{F}_T)$, which turns out to represent the nonlinear expectations

$$\mathcal{E}^g(X|\mathcal{F}_t) = Y_t, \qquad X \in L^2(\mathcal{F}_T),\ t \in [0, T].$$

Depending on the properties of g, we have $\rho_t(X) = \mathcal{E}^g(-X|\mathcal{F}_t)$.

[3] Peng (1997, 2003), Frittelli, Rosazza Gianin (2002, 2004), Rosazza Gianin (2006), ...

The properties of the driver g characterise the properties of $(Y_t)_t$.
For instance,

- When the driver is assumed Lipschitz, we have guarantee of existence and the unicity of the solution.
- Beyond this case (e.g. quadratic), one can study concepts of "maximal solutions"[4].
- If $g(t, 0, 0) = 0$, then normalisation is guaranteed.
- Properties of convexity of g in the couple (y, z) provide convex solutions.
- When g does not depend on Y, then the $\mathcal{F}_t$-translation invariance is satisfied[5]

Other notices of interest

- Having some dynamics, we wish to consider numerical computation techniques (...)
- The future may not be Gaussian, other family of noises considered[6]

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[4] Kobilanski (2000) and also Barrieu and El Karoui (2009

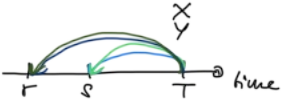
[5] Barrieu and El Karoui (2009), Jiang (2008).

[6] Royer (2006), Quenez, Sulem (2013), Laeven, Stadje(2014, DiNunno, Sjursen (2014), Sulem, Øksendal (2019),...

Dynamic risk meas: time consistency?

Dealing with phenomena in time, also risk assessment has to follow.

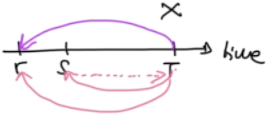
We take different times of evaluation: $r \leq s \leq T$



Order time-consistency: For $X, Y \in \mathcal{X}_T$,

$$\rho_s(X) = \rho_s(Y) \quad \Longrightarrow \quad \rho_r(X) = \rho_r(Y).$$

Strong time-consistency: For $X \in \mathcal{X}_T$,



$$\rho_r(X) = \rho_r(-\rho_s(X)).$$

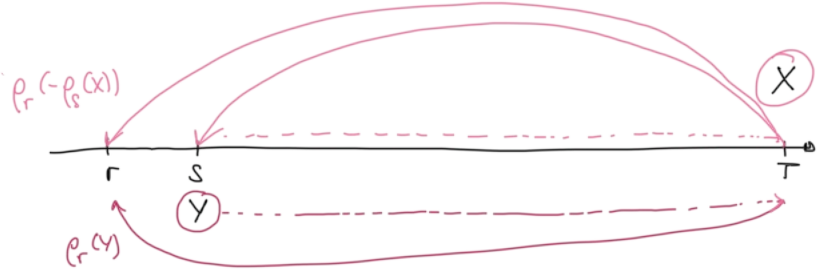
Careful!!
Static risk measures with different time-zones are time-inconsistent[7].

Result:
If a dynamic risk meas. is normalised, then the two concepts are equivalent.

[7] Examples by Artzner, Cheredito, Delaben, Föllmer, Cohen, Stadje '06-10. Modification and study of order time consistency: Bion-Nadal, Detlefsen, Scandolo, Delbaen, Bielechi, Cialenko '08-'10

Horizon and time-consistency, problems?

Let's go deeper into "strong time-consistency"



Then horizon risk emerges connected to the use of the wrong risk measure for the targeted horizon.

We then introduce the concepts of fully dynamic risk meas. and the restriction property, and we quantify horizon risk by the horizon-longevity [8]

[8] Bion-Nadal, DiNunno (2020), DiNunno, Rosazza Gianin (2024)

Fully-dynamic risk measures

Fully-dynamic (convex) risk measure[9] is a *family* $(\rho_{st})_{s,t}$ of risk measures :

$$\rho_{st} : \mathcal{X}_t \longrightarrow \mathcal{X}_s$$

In many applications, we consider each of the risk meas. satisfying

- monotonicity, convexity
- $\mathcal{F}_s$-translation invariance or cash additivity , i.e. for $X \in L^p(\mathcal{F}_t)$,

$$\rho_{st}(X + m) = \rho_{st}(X) - m, \text{ for all } m \in L^p(\mathcal{F}_s)$$

The acceptance set of ρ_{st} is defined as $\mathcal{A}_{st} \triangleq \{Z \in \mathcal{X}_t : \quad \rho_{st}(Z) \leq 0 \ P\text{-a.s.}\}$.
Considering the setup of random variables with moments $L^p(P)$, then each risk measure admits the dual representation

$$\rho_{st}(X) \quad = \quad \operatorname*{ess\,max}_{Q \in \mathcal{Q}_{st}} \{E_Q[-X|\mathcal{F}_s] - \alpha_{st}(Q)\}$$

here α_{st} is *minimal penalty* and $\mathcal{Q}_{st} = \{Q \text{ on } \mathcal{F}_t : Q \ll P \text{ and } Q|_{\mathcal{F}_s} \equiv P|_{\mathcal{F}_s}\}$.

[9] Bion-Nadal, DiNunno (2020)

Comments

- We do not assume a priori that the risk measures ρ_{st} are normalised, i.e.

$$\rho_{st}(0) = 0, \quad \text{for all } s \leq t,$$

- We do not assume that the risk measures have the restriction property, i.e.

$$\rho_{rs}(Y) = \rho_{rt}(Y), \qquad \text{for all } Y \in L^p(\mathcal{F}_s), \qquad r \leq s \leq t$$

Remark: relationship with dynamic risk measures

A *fully-dynamic risk measure with restriction property corresponds one-to-one with a dynamic risk measure*:

$$\rho_r(Y) = \rho_{rT}(Y) = \rho_{rs}(Y), \qquad \text{for all } Y \in L^p(\mathcal{F}_s), \qquad r \leq s \leq T.$$

$(\rho_{st})_{s,t}$ and time-consistency

Going back to the analysis of time-consistency, we now have

Definition. A fully-dynamic risk measure $(\rho_{st})_{s,t}$ is

- order time-consistent if for $r \leq s \leq t$, $X, Y \in \mathcal{X}_t$, we have
$$\rho_{st}(X) = \rho_{st}(Y) \quad \Longrightarrow \quad \rho_{rt}(X) = \rho_{rt}(Y).$$
- weak time-consistent, if for $r \leq s \leq t$, $X \in \mathcal{X}_t$,
$$\rho_{rt}(X) = \rho_{rt}(\rho_{st}(0) - \rho_{st}(X))$$
- recursive if for $r \leq s \leq t$, we have
$$\rho_{rt}(X) = \rho_{rs}(-\rho_{st}(X)), \quad X \in \mathcal{X}_t,$$

See Acciaio, Penner (2011), Bielecki, Cialenco, Pitera (2017), Bion-Nadal (2009),...
Weak time consistency appeared in Bion-Nadal, DiNunno (2020) in relation to risk indifference prices.

About recursive time-consistency

- Recursivity is a “composition rule”.

- Recursivity <u>is not</u> transferred via normalisation.

If $(\rho_{st})_{s,t}$ is strong time consistent, then its normalised version

$$\bar{\rho}_{st}(X) := \rho_{st}(X) - \rho_{st}(0), \quad X \in \mathcal{X}_t, \quad s \le t,$$

may not be.

Indeed, the values $\rho_{rt}(0)$, $\rho_{rs}(0)$, and $\rho_{st}(0)$ are potentially different.

Remark

A normalised fully-dynamic risk measure $\bar{\rho}_{st}(X) := \rho_{st}(X) - \rho_{st}(0)$, is recursive if and only if

$$\rho_{rt}(0) = \rho_{rs}(0) + E_Q\left[\rho_{st}(0)|\mathcal{F}_r\right]$$

About order time-consistency

- Order time-consistency is transferred to the normalised fully-dynamic risk measures.

Proposition. The following statements are equivalent:

i) $(\rho_{st})_{s,t}$ is *recursive*.

ii) $(\rho_{st})_{s,t}$ is *order time-consistent* and

$$\rho_{rt}(Y) = \rho_{rs}(Y - \rho_{st}(0)), \quad 0 \leq r \leq s \leq t, \ \ Y \in \mathcal{X}_s.$$

Corollary. If the fully-dynamic risk measure is normalised, then we have the equivalence:

i) $(\rho_{st})_{s,t}$ is *recursive*

ii) $(\rho_{st})_{s,t}$ is order time-consistent and the restriction property holds.

Corollary for dynamic risk measures. If $(\rho_s)_s$ is normalised, then the following two are equivalent:

i) order time-consistency

ii) $\rho_r(X) = \rho_r(-\rho_s(X)), \quad X \in L_p(\mathcal{F}_t),$

About weak time-consistency

Proposition. For fully-dynamic risk measures we have equivalence

i) weak time-consistency

ii) order time-consistent.

(Here cash additivity is crucial!)

Remark: Under both normalisation and restriction, all the three concepts coincide.

Side note: We can characterise the concepts in terms of minimal penalties in the dual representation.

About weak time-consistency

Proposition. For fully-dynamic risk measures we have equivalence

i) weak time-consistency

ii) order time-consistent.

(Here cash additivity is crucial!)

Remark: Under both normalisation and restriction, all the three concepts coincide.

Side note: We can characterise the concepts in terms of minimal penalties in the dual representation.

Take home message:
Normalisation and restriction are crucial characteristics in dynamic risk-evaluation.
Assuming these becomes a modelling choice, which should not be underestimated!

Horizon risk and H-longevity

Once we drop restriction, to allow for the evaluation of horizon risk, we introduce h-longevity as a kind of penalisation for using a risk measure non-appropriate for the time window.

Definition. Horizon longevity or h-longevity is

$$\gamma(s, t, u, X) := \rho_{su}(X) - \rho_{st}(X) \geq 0$$

for any $0 \leq t \leq u, X \in \mathcal{X}_t$.

Proposition (acceptance sets). For a fully-dynamic risk measure $(\rho_{st})_{s,t}$:

(a) H-longevity is equivalent to $\mathcal{A}_{su} \cap \mathcal{X}_t \subseteq \mathcal{A}_{st}$ for any $s \leq t \leq u$.
(b) Restriction is equivalent to $\mathcal{A}_{su} \cap \mathcal{X}_t = \mathcal{A}_{st}$ for any $s \leq t \leq u$.

$(\rho_{st})_{s,t}$ generated by one BSDE

Focus on $L^2(P)$-spaces and a d-dimensional Brownian noise $(B_t)_t$.
The process $(Y_s)_s$ of the solution $(Y_s, Z_s)_{s\in[0,t]}$ to the BSDE

$$Y_s = X + \int_s^t g(r, Z_r)\, dr - \int_s^t Z_r\, dB_r = \mathcal{E}^g(X|\mathcal{F}_s)$$

with a convex driver g not depending on y, represents the nonlinear expectation and the risk measure:

$$\rho_{st}(X) = \mathcal{E}^g(-X|\mathcal{F}_s), \qquad X \in L^2(\mathcal{F}_t).$$

Proposition. The following properties are equivalent:

- $g(t, 0) = 0$ for any $t \in [0, T]$;
- each ρ_{st} is normalised
- $(\rho_{st})_{s,t}$ satisfies the restriction property

Hence, $(\rho_{st})_{s,t}$ generated from the BSDE with $g(t, 0) = 0$ are recursive.

The quantification of h-longevity can be retrieved.

Proposition. If $g(v, 0) \geq 0$ for any v, then h-longevity holds. Furthermore,

$$\gamma(s, t, u, X) = E_{\widetilde{Q}_X}\left[\int_t^u g(v, 0)dv | \mathcal{F}_s\right],$$

where $\widetilde{Q}_X \sim P$ is a suitable probability measure depending on X.

$(\rho_{st})_{s,t}$ generated by a family of BSDEs

To give even more emphasis to the time horizon, we induce risk measures from *a family of BSDEs with convex drivers* $\mathcal{G} = (g_t)_t$,*depending on the time horizon t* in the form

$$Y_s = X + \int_s^t g_t(r, Z_r)\, dr - \int_s^t Z_r\, dB_r$$

Then we have $\rho_{st}(X) = \rho^{\mathcal{G}}_{st}(X) = \mathcal{E}^{g_t}(-X|\mathcal{F}_s),$ for any $X \in L^2(\mathcal{F}_t)$.

NB: If, for all t, $g_t(r, 0) = 0$ for any r, then $\rho^{\mathcal{G}}_{st}$ is normalised. However, this does NOT imply the restriction property.

Proposition

Whenever $g_t(r, 0) = 0$, for any r, t, with $g_t(r, \cdot)$ be continuous in r.
The restriction property holds if and only if g_t is constant in u
(i.e. back to a single BSDE!).

Example. Consider the driver $gt(r, z) \equiv a_t \in \mathbb{R} \setminus \{0\}$. Then

$$\rho_{st}(X) = E_P\left[-X|\,\mathcal{F}_s\right] + (t - s)a_t.$$

$(\rho_{st})_{s,t}$ is NOT normalised and does NOT satisfy the restriction property.

When it comes to h-longevity, we have the following result.

Proposition

i) If $\mathcal{G}$ is increasing and $\rho_{tu}(0) \geq 0$ for any $t \leq u$, then $(\rho_{tu})_{t,u}$ satisfies h-longevity.

ii) If $\mathcal{G}$ is increasing and $g_t \geq 0$ for any $t \in [0, T]$, then $(\rho_{tu})_{t,u}$ satisfies h-longevity.

In fact, this result relies on

Theorem: comparison of BSDEs on different horizons $[0, T_1] \subset [0, T_2]$

Consider two BSDEs:

$$Y_s^{T_i} = \xi_i + \int_s^{T_i} g^{T_i}(r, Y_r^{T_i}, Z_r^{T_i})dr - \int_s^{T_i} Z_r^{T_i} dB_r.$$

We obtain that $Y_s^{T_2} \geq Y_s^{T_1}$ for any $s \in [0, T_1]$ and $Y_s^{T_2} \geq \xi_1$ for any $s \in [T_1, T_2]$, whenever

- $g^{T_2}(r, y, z) \geq g^{T_1}(r, y, z)$ for any $r \in [0, T_1], y, z$
- $g^{T_2}(r, y, z) \geq 0$ for any $r \in [T_1, T_2], y, z$
- $\xi_2 \geq \xi_1$.

Examples

(a) Consider now the driver $g_t(r,z) = bz + a_t$ with $a_t, b \in \mathbb{R} \setminus \{0\}$ and a_t depending on the maturity t.
It follows that

$$\rho_{st}(X) = E_Q\left[-X \middle| \mathcal{F}_s\right] + (t-s)a_t,$$

where $E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] = \exp\left\{-\frac{1}{2}b^2 t + b \cdot B_t\right\}$. For $a_t \neq 0$, $(\rho_{st})_{s,t}$ is NOT normalised and does NOT satisfy the restriction property. Instead, it satisfies H-longevity whenever $a_t > 0$ is increasing in t.

b) Entropic type risk measures (quadratic BSDEs)

In the one-dimensional case, consider

$$Y_s = -X + \int_s^t \left[b_t \frac{Z_r^2}{2} + a_t\right] dr - \int_s^t Z_r dB_r$$

with b_t and a_t positive functions. Then we have

$$\rho_{st}(X) = \frac{1}{b_t} \ln\left(E_P\left[\exp(-b_t X) \middle| \mathcal{F}_s\right]\right) + \int_s^t a_t(r)dr.$$

Hence, $(\rho_{st})_{s,t}$ is NOT normalized and does NOT satisfy the restriction property. Instead, it satisfies h-longevity whenever $(a_t)_t$ and $(b_t)_t$ are increasing in t.

D-tour: $(\rho_{st})_{s,t}$ generated by BSVIE

Exploration on the use of BSVIE[10] to generate fully dynamic risk measures[11]. We consider

$$Y_s = X + \int_s^T g(s, r, Z(s, r))\, dr - \int_s^T Z(s, r)\, dB_r,$$

where the driver is

$$g : \Omega \times \Delta \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$$

with $\Delta \triangleq \{(s, r) \in [0, T] \times [0, T] : s \leq r\}$

Relationship with a family of BSDEs (parametrised by v):

$$\eta(s; v, X) = X + \int_s^T g(v, r, Z(v, r))\, dr - \int_s^T Z(v, r)\, dB_r, \quad v \in [s, T]$$

then $\quad Y_s = \eta(s; s, X)$

[10] See Yong (2007)

[11] See DiNunno, Rosazza Gianin (2024) with convex representation and converse comparison theorems

Proposition: h-longevity.

If $g(s,v,0) \geq 0$ for any $s \leq v$, then longevity holds.
Furthermore,

$$\gamma(s,t,u,X) = E_{\widetilde{Q}_{s,X}}\left[\int_t^u g(s,v,0)dv \Big| \mathcal{F}_s\right], \qquad s \leq t \leq u,$$

where $\widetilde{Q}_{s,X \sim P}$ is a suitable probability measure depending on X.

Example revisited: Entropic type risk measures.

Consider

$$Y_s \;=\; -X + \int_s^T a(s,r)dr + \int_s^T b(s)\frac{(Z(s,r))^2}{2}dr - \int_s^T Z(s,r)dB_r^{\widetilde{Q}_s}$$

with positive deterministic functions b and a.
Then

$$Y_s = \frac{1}{b(s)} \ln E_P\left[e^{-b(s)X} \Big| \mathcal{F}_s\right] + \int_s^T a(s,r)dr,$$

that is a translation of the usual entropic risk measure.
Choosing $a(s,r) > 0$, there is h-longevity.

LONG horizons, money, interest rates

The value of money varies over long time horizons, then we also have uncertainty on interest rates.

The combination of both longevity and interest rates, we enter the domain of cash non-additive risk measure.

Quantities expressed in unit of money and $€_t$ is the unit of money at time t. Hence a financial investment available at time t is denoted $X€_t$, where X represents the size of the investment.

Let $(D_{st})_{0 \leq s \leq t \leq T}$ be the family of discount factors D_{st} on the time interval $(s, t]$:

$$0 < d_{st} \leq D_{st} €_t \leq 1.$$

The unit of measurement for D_{st} is $1/€_t$.

For any cash additive fully-dynamic risk measure $(\varphi_{st})_{0\leq s\leq t\leq T}$ we define

$$\rho_{st}(X) \triangleq \varphi_{st}(D_{st}X€_t), \qquad X \in L^p(\mathcal{F}_t).$$

Indeed, ρ_{st} is cash subadditive. For any $X \in L^p(\mathcal{F}_t)$ and $m \in L^p_+(\mathcal{F}_s)$, we have

$$\begin{aligned}\rho_{st}(X+m) &= \varphi_{st}(D_{st}(X+m)€_t) \\ &\geq \varphi_{st}(D_{st}X€_t + m€_t) = \varphi_{st}(D_{st}X€_t) - m \\ &= \rho_{st}(X) - m,\end{aligned}$$

thanks to monotonicity.

Another cash subaddtive risk measure generated by the ambiguity of the interest rates is given by:

$$\mathcal{R}_{st}(X) \triangleq \operatorname*{ess\,sup}_{0<d_{st}\leq D_{st}€_t\leq 1} \varphi_{st}(D_{st}X€_t)$$

In the framework of cash non-additive risk measures we can study h-longevity, normalisation, and time-consistency.

Cash non-additive $(\rho_{st})_{s,t}$ and BSDE

In a dynamic setting, we can generate cash non-additive risk measures from BSDEs with explicit dependence on Y in the driver:

$$Y_t = X + \int_t^u g(s, Y_s, Z_s)\, ds - \int_t^u Z_s\, dB_s$$

In particular we have that:

- if $g(s, y, z)$ is decreasing in y for all (s, z), then the risk measure generated ρ_{tu} is cash sub-additive[12].

Proposition.

- ρ_{tu} is normalised if and only if $g(t, 0, 0) = 0$ for all t.
- ρ_{tu} is restricted (and normalised) if and only if $g(t, y, 0) = 0$ for all t, y.

[12] See El Karoui, Ravanelli 2009)

Cash non-additive $(\rho_{st})_{s,t}$ and BSDE

Proposition.
a) recursivity implies oder time-consistency.
b) Weak time-consistency implies order time-consistency.
c) Under (normalization and) restriction: recursive is equivalent to weak time-consistency.

N.B. Order time-consistency does not imply weak time-consistency!

Example. Consider

$$\rho_{tu}(X) = E_P\big[-e^{-r(u-t)}X\big|\mathcal{F}_t\big], \quad X \in L^P(\mathcal{F}_u),$$

with $r > 0$. Then $(\rho_{tu})_{t,u}$ is a cash subadditive and normalized fully-dynamic risk measure that satisfies recursive and order time-consistency. Nevertheless, weak time-consistency does not hold. In fact,

$$\begin{aligned}\rho_{su}(\rho_{tu}(0) - \rho_{tu}(X)) &= \rho_{su}(-\rho_{tu}(X)) \\ &= e^{-r(u-t)}\rho_{su}(X) \neq \rho_{su}(X).\end{aligned}$$

Cash non-additive $(\rho_{st})_{s,t}$ and h-longevity

Proposition.
H-longevity holds if and only if $g(t, y, 0) \geq 0$ for any t , y. Furthermore, for $s, u \in [0, T]$ with $s \leq u$, we have

$$\gamma(s, t, u, X) = E_{\widetilde{Q}_X}\Big[e^{\int_s^u \Delta_y g(v) dv} \int_t^u g(v, -X, 0) dv | \mathcal{F}_s\Big], \quad s \leq t \leq u, X \in L^p(\mathcal{F}_t),$$

where $\widetilde{Q}_X \sim P$ is a suitable probability measure depending on X.

Example: q-entropic risk measures

Here we are driven by considerations on capital requirements on potential losses in long term horizons.

$$Y_t = -X + \int_t^u \left[\frac{q}{2} \frac{Z_s^2}{1 + (1-q)Y_s} + a(s) \right] ds - \int_t^T Z_s dB_s$$

The solution of such BSDE is

$$Y_t = \ln_q E\Big[\exp_q \Big(-X + \int_t^u a(s)ds \Big) \Big| \mathcal{F}_t \Big]$$

given in terms of the generalised q-exponential and q-logarithmic functions, for $q > 1$ or $q \in (0, 1)$:

$$\exp_q(x) = [1 - (1-q)x]^{\frac{1}{1-q}} \qquad \ln_q(x) = \frac{x^{1-q} - 1}{1-q}$$

with split domain depending on q:

$$\begin{cases} q \in (0,1), \quad Dom(\exp_q):\ x \geq -\frac{1}{1-q}; \quad Dom(ln_q):\ x \geq 0 \\ q > 1, \quad Dom(\exp_q):\ x < -\frac{1}{1-q}; \quad Dom(ln_q):\ x > 0 \end{cases}$$

q-entropic measure on losses

Call $(\varphi_{tu})_{t,u}$ the solution of the BSDE above in the case $q \in (0,1)$:

$$\varphi_{tu}(X) = Y_t.$$

Then we can defined:

$$\rho_{tu}^{q,a}(X) \triangleq \varphi_{tu}\big(-(X+\beta)^+\big), \qquad X \in L^2(\mathcal{F}_u),$$

where β represents a level of acceptable loss. Then

$$\rho_{tu}^{q,a}(X) = \ln_q E_P\left[\exp_q\left((X+\beta)^- + \int_t^u a(s)ds\right)\bigg|\,\mathcal{F}_t\right],$$

This risk measure is convex, cash subadditive, not normalised, not restricted, and there is h-longevity whenever $a(s) > 0$.

Proposition (comparison among entropics) Take $a \equiv 0$. For any $X \in L^2(\mathcal{F}_u), \beta \in \mathbb{R}$, the q-entropic risk measure on losses ρ_{tu}^q is increasing in q with

$$E_P[-(X+\beta)^-|\mathcal{F}_t] = \rho_{tu}^0(X) \le \rho_{tu}^q(X) \le \rho_{tu}^1(X) = \rho_{tu}^{entr}(-(X+\beta)^-).$$

Summing up

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- It also requires that time scales are considered to avoid horizon risk, this is also a modelling need, which requires specific attention
- when horizon are long, other elements can come into play in the robustness of the model, coordination with other uncertainties (see example of interest rates)

Summing up

- Dynamic risk measurement needs time consistency in evaluation time and this implies modelling needs and it is treated accordingly (see normalisation and restriction)
- It also requires that time scales are considered to avoid horizon risk, this is also a modelling need, which requires specific attention
- when horizon are long, other elements can come into play in the robustness of the model, coordination with other uncertainties (see example of interest rates)
- A word about numerics ....

Numerical methods

- Numerics for BSDEs often does not consider without risk evaluation: assumptions are too strong, e.g., no methods for quadratic case and unbounded risks.
- Computationally, there is a different between computing
  1. $\rho_{st}(X)$ for a given X and
  2. $\rho_{st}(\cdot)$.

  Here, note that X is often the forward S(P)DE of a phenomena:

$$dX_t = \beta(t, X_t)dt + \sigma(t, X_t)dB_t$$

  Case (i) is typically dealt with Forward-Backward SDEs in a system.
- An operator valued argument to obtain (ii) is based on Wiener-chaos expansions coupled with more classical numerical methods for BSDEs. We obtain the Operator Euler Scheme for BSDEs[13].

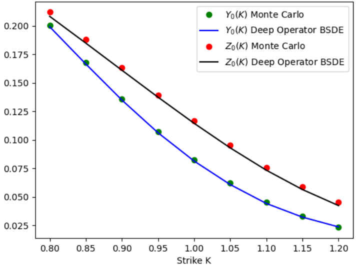
[13] DiNunno, Diaz (2024+)

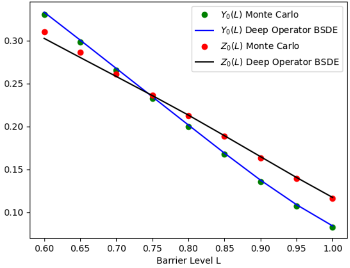
Example: barrier options

Potential risk: $X(K, L) := (S_T - K)_+ \mathbf{1}_{\{S_{t_i} \geq L \ \forall i \in \{0,\ldots,n\}\}}$, where S follows a Black-Scholes diffusion

$$S_t = s_0 e^{(\mu - (1/2)\sigma^2)t + \sigma B_t}, \quad \forall t \in [0, T].$$

dynamic risk measure: with driver $g(t, y, z) = -ry - z(\mu - r)/\sigma$
range: $K \in [0.8, 1.2], L = 0.85$ and $L \in [0.6, 1], K = 0.95$



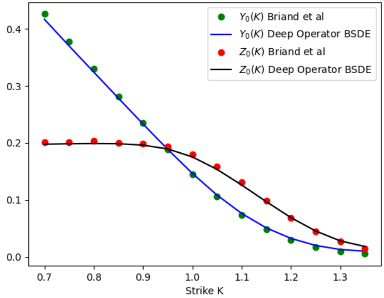


Example: asian option

Potential risk: $\xi(K) := \left(\sum_{i=0}^{n-1} \Delta t_i S_{t_i} - K\right)^+$.
Dynamic risk measure:

$$g(t,y,z) = -ry - \frac{\mu - r}{\sigma} z + (R-r)\Big(y - \frac{z}{\sigma}\Big)-.$$

range: $K \in [0.7, 1.35]$



THANK YOU FOR YOUR ATTENTION