TFS symposium

Homogenization of local and nonlocal operators and homogeneous geometric structures

UiT campus Narvik E2463, 31 March – 1 April 2025

Supported by **TFS** project **Pure Mathematics in Norway**

This meeting aims to bring together people working on modern homogenization theory for PDEs and nonlocal operators of convolution type as well as homogeneous differential geometric structures. Among the models of interest are homogenization of spectral problems for high contrast operators in periodic and random media, homogenization combined with dimension reduction, singularly perturbed operators with oscillating coefficients, as well as classification of highly symmetric models of geometries and PDEs.

Organizers:

Boris Kruglikov (UiT, Tromsø), Andrei Piatnitski (UiT, Narvik), Elena Zhizhina (UiT, Narvik)

Participants:

Michail Cherdantsev (University of Cardiff), Shane Cooper (UCL, London), Subbarao Venkatesh Guggilam (UiT, Tromsø), Ilia Kamotski (UCL, London), Irina Markina (University of Bergen), Wijnand Steneker (UiT, Tromsø), Valery Smyshlyaev (UCL, London), Dennis The (UiT, Tromsø), Igor Velcic (University of Zagreb), Henrik Winther (UiT, Tromsø), Rodolfo Antonio Rios Zertuche (UiT, Tromsø)

Program 31 March

12.45	*Lunch UiT cafeteria*	
13.00	*Valery Smyshlyaev (UCL, London)*	Uniform spectral asymptotics for high-contrast periodic media
13.45	*Irina Markina (University of Bergen)*	The fundamental solution of a class of ultra-hyperbolic operators on pseudo-H-type Lie groups
14.30	*Rodolfo Antonio Rios Zertuche (UiT, Tromsø)*	Higher-dimensional weak KAM theory
15.15	*Coffee break*	
15.45	*Igor Velčić (University of Zagreb, Croatia)*	Poroelastic plate model obtained by simultaneous homogenization and dimension reduction
16.30	*Mikhail Cherdantsev (University of Cardiff)*	Homogenisation of high-contrast convolution-type integral operators
17.15	*Wijnand Steneker (UiT, Tromsø)*	Variationality of conformal geodesics in dimension 3
19:30	*Conference Dinner*	

Program 1 April

09.00	*Dennis The (UiT, Tromsø)*	Two-scale type operator and spectral approximations and estimates for a class of asymptotically degenerating problems
09.45	*Ilia Kamotski (UCL, London)*	Two-scale type operator and spectral approximations and estimates for a class of asymptotically degenerating problems
10:30	*Coffee break*	
11.00	*Subbarao Venkatesh Guggilam (UiT, Tromsø)*	Chen–Fliess Series: An Interplay between Algebraic and Geometric Aspects of Systems Theory.
11.45	*Shane Cooper (UCL, London)*	Quantitative homogenisation for differential equations with highly anisotropic partially degenerating coefficients
12:30	*Lunch UiT cafeteria*	

Abstracts

Mikhail Cherdantsev (University of Cardiff):
Homogenisation of high-contrast convolution-type integral operators.

We consider homogenisation problem for convolution type integral operators in a periodic high-contrast medium comprising of "stiff" and "soft" components with the so-called double porosity type scaling between the components' contrast and the period, while the convolution kernel is subject to a diffusive scaling. We show that the limit operator is of two-scale nature. The 'macroscopic' part of this two-scale operator, arising from the stiff component, is an elliptic second order differential operator with constant (homogenised) coefficients. The 'microscopic' part of the limit operator comes from the soft component.

We then analyse the limiting spectrum and its relation to the spectrum of the limit two-scale operator. Remarkably, the limiting spectrum is, in general, strictly larger than the spectrum of the two-scale limit operator. This is due to the fact that the two-scale convergence 'eliminates' in the limit all non-periodic modes pertinent to the soft component. We provide a characterisation of the limiting spectrum and discuss the effect of the boundary layer. Finally, in the whole space setting we obtain norm-resolvent convergence via scaled Gelfand transform using a novel approach of Cooper-Kamotski-Smyshlyaev.

This a joint work with Andrei Piantnitski (Narvik) and Igor Velcic (Zagreb).

Shane Cooper (UCL, London): *Quantitative homogenisation for differential equations with highly anisotropic partially degenerating coefficients*.

We consider a non-uniformly elliptic second-order differential operator with periodic coefficients that models composite media consisting of highly anisotropic cylindrical fibres periodically distributed in an isotropic background. The degree of anisotropy is related to the period of the coefficients via a `critical' high-contrast scaling. In particular, ellipticity is lost in certain directions as the period ε tends to zero. Our primary interest is in the asymptotic behaviour of the resolvent of this operator in the limit of small ε.

Two-scale resolvent convergence results were established for such operators in Cherednichenko, Smyshlyaev and Zhikov: Proc. Royal Soc. Edinburgh: Sect. A Mathematics 136 (1), 87—114 (2006). In this work, we provide an asymptotic description of the resolvent and establish operator-type error estimates. Our approach adopts the general scheme of Cooper, Kamotski and Smyshlyaev (preprint arXiv:2307.13151). However, we face new challenges such as a directional dependence on the loss of ellipticity in addition to a key `spectral gap'

assumption of the above article only holding in a weaker sense. This results in an additional "interfacial" boundary layer analysis in the vicinity of each fibre to arrive at order ε operator-type error estimates.

This is joint work with Ilia Kamotski.

Subbarao Venkatesh Guggilam (UiT, Tromsø):
Chen–Fliess Series: An Interplay between Algebraic and Geometric Aspects of Systems Theory.

Chen--Fliess series is a formal power series (functional series) of iterated integrals of a (control) path $\mathrm{u}\colon [0,\mathrm{T}] \to \mathbf{R}^{\mathrm{m}}$ is the input-output operator of the formal differential equation, termed as "input-affine" differential equation given by

$$\mathrm{z}' = \mathrm{f}_0(\mathrm{z}) + \sum_{1\le \mathrm{i}\le \mathrm{m}} \mathrm{f}_\mathrm{i}(\mathrm{z})\mathrm{u}_\mathrm{i}(\mathrm{t})\,, \mathrm{z}(0) = \mathrm{z}_0 \in \mathbf{R}^{\mathrm{n}}, \qquad \mathrm{y}(\mathrm{t}) = \mathrm{h}\big(\mathrm{z}(\mathrm{t})\big) \in \mathbf{R}^{\mathrm{p}}.$$

The vector fields $f_0, \dots, f_m$ are generators of free Lie algebra in $\mathrm{m}+1$ letters. The Chen--Fliess series, indexed by a noncommutative formal power series say *c* is denoted by F_c is a map from control path *u* to the output $\mathrm{y}(\mathrm{t})$. The Chen--Fliess framework is coordinate-free and thus any computations and statements about the system properties are intrinsic. The talk will start with a rudimentary introduction to the Chen--Fliess series and how the Chen--Fliess series are algebraically closed by the various interconnection products.

The talk will look into a specific intrinsic property called "Relative Degree" and the computation of relative degree under various interconnections of two Chen--Fliess series. If time permits, the talk will proceed towards a specific case called affine feedback interconnection and how the feedback product is computed via combinatorial Hopf algebra of coordinate functions on the feedback group and linearizing (projecting onto the primitives) the coproduct and then taking the graded dual gives us a post--Lie algebra. The talks is based primarily on the following works:

[1] K. Ebrahimi-Fard, W.S. Gray, G.S. Venkatesh, *On the Post–Lie Structure in SISO Affine Feedback Control Systems*, arXiv:2311.04070 [math.OC].

[2] W. S. Gray, G. S. Venkatesh, *Relative Degree of Interconnected SISO Nonlinear Control Systems*, Systems and Control Letters 124, 99–105 (2019).

Ilia Kamotski (UCL, London): *Two-scale type operator and spectral approximations and estimates for a class of asymptotically degenerating problems*.

We review an abstract scheme [*] allowing to construct two-scale type approximations for a wide class of asymptotically degenerating operators and their spectra. This is accompanied by operator estimates, and illustrated by various examples. Some most recent developments on improved error estimates will also be discussed.

[*] S. Cooper, I.V. Kamotski, V.P. Smyshlyaev, *Quantitative multiscale operator-type approximations for asymptotically degenerating spectral problems*. arXiv:2307.13151 (2025).

Irina Markina (University of Bergen): *The fundamental solution of a class of ultra-hyperbolic operators on pseudo-H-type Lie groups*.

Pseudo-H-type Lie groups $N_{r,s}$of signature *(r,s)* are defined via a module action of the Clifford algebra $Cl_{r,s}$ on a vector space $\mathrm{V} \cong \mathbf{R}^{2\mathrm{n}}$. They form a subclass of all 2-step nilpotent Lie groups and based on their algebraic structure they can be equipped with a left-invariant pseudo-Riemannian metric. Let $n_{r,s}$ denote the Lie algebra corresponding to $N_{r,s}$. A choice of left-invariant vector fields $X_1, \dots, X_{2n}$ which Lie generate a complement of the center of $n_{r,s}$ gives rise to a second order operator, which we call ultra-hyperbolic:

$$\Delta_{r,s} = X_1^2 + \cdots + X_n^2 - X_{n+1}^2 - \cdots - X_{2n}^2.$$

In terms of classical special functions we present families of fundamental solutions of $\Delta_{r,s}$ in the case $r = 0, s > 0$ and study their properties. In the case of $r > 0$ the operator $\Delta_{r,s}$ admits no fundamental solution in the space of tempered distributions.

This is a joint work with W. Bauer and A. Froehly (Leibniz Universität Hannover, Germany).

Rodolfo Antonio Rios Zertuche (UiT, Tromsø): *Higher-dimensional weak KAM theory*.

Weak KAM emerged in the 90s as a machinery to obtain viscosity solutions to the Hamilton-Jacobi equation, in the context of Lagrangian dynamics. We develop an axiomatization of the theory in category-theoretical language, and we apply it to give a new understanding of variational problems involving higher-dimensional manifolds, thus generalizing the original situation in which only curves were involved.

Valery Smyshlyaev (UCL, London):
Uniform spectral asymptotics for high-contrast periodic media.

For a "double-porosity'' type elliptic operator with high-contrast periodic matrix-inclusion coefficients in $\mathbf{R}^n$, $n \geq 2$, we obtain improved results on uniform approximations of typical Floquet-Bloch eigenvalues in terms of those of Zhikov's two-scale limit operator. As a result, we obtain not only improved rates for convergence of the spectra to the limit spectrum, that contains band gaps, but also improved uniform estimates for an explicit asymptotics for the integrated density of states.

This is a joint work with Ilia Kamotski and Shane Cooper.

Dennis The (UiT, Tromsø):
Homogeneous CR real hypersurfaces in $\mathbf{C}^3$ and associated PDE systems.

Holomorphically (locally) homogeneous CR real hypersurfaces M^3 in $\mathbf{C}^2$ were classified by Élie Cartan in 1932. The corresponding classification of M^5 in $\mathbf{C}^3$ was finally completed in 2020, with contributions from a number of authors. I will survey this story, describing in particular the role that 2nd order PDE systems played in completing the classification problem.

Based on joint works with Boris Doubrov, Alexander Medvedev and Joel Merker.

Wijnand Sebastiaan Steneker (UiT, Tromsø):
Variationality of conformal geodesics in dimension three.

Conformal geodesics form an invariantly defined family of unparametrized curves in a conformal manifold generalizing unparametrized geodesics/paths of projective connections. The equation describing them is of third order, and it was an open problem whether they are given by an Euler--Lagrange equation. In dimension 3 (the simplest, but most important from the viewpoint of physical applications) we demonstrate that the equation for unparametrized conformal geodesics is variational. The homogeneous case, namely that of the conformally flat space was known before, with the extremals being circles, and we extend its geometric interpretation.

This is a joint work with Boris Kruglikov and Vladimir S. Matveev.

Igor Velčić (University of Zagreb, Croatia):
Poroelastic plate model obtained by simultaneous homogenization and dimension reduction.

Starting from 3d linear model of fluid- elastic structure interaction and by doing simultaneous homogenization and dimension reduction we derive the model of Biot's plate under the assumption that the size of pores is much smaller than the thickness of the body. We also discuss the model of contact of two poroelastic plates.

This is a joint work with Marin Buzančić, Pedro Hernandez Llanos and Josi Žubrinić.